USE OF THE LEONTOVICH BOUNDARY CONDITION IN THE CALCULATION OF ELECTROMAGNETIC RADIATION FROM A GAMMA-RAY SOURCE

L. P. Gorbachev and A. Yu. Matronchik UDC 537.812

1. A time-dependent gamma-ray source on the boundary between air and a conducting halfspace excites radio-frequency electromagnetic waves in the surrounding space. The problem of calculating the parameters of this field has been considered repeatedly in the literature (see the review in [1]). It is well known [1-4] that the radiation mechanism reduces to the following fundamental processes: Gamma-rays are inelastically scattered by the molecules of the medium (in this case air) and create a current of Compton electrons. The electrons propagate mainly in the radial direction from the source and ionize the air, and a back conduction current results. The resulting system of electric currents generates a time-dependent electromagnetic field, whose parameters are determined by the characteristics of the gamma-ray source and the surrounding medium. In particular, an important factor is the electrical characteristics of the surface such as its relative dielectric permittivity ε_0 and electrical conductivity σ_0.

A numerical method of solving the problem was worked out in [2] under the assumption that the conductivity of the surface is infinite ($\sigma_0 = \infty$). In this case, the tangential component of the electric field on the boundary of the conducting halfspace is zero and the angular dependence in Maxwell's equations separates out. Thus the two-dimensional problem can be reduced to a set of one-dimensional time-dependent problems.

In [3, 4] an attempt was made to take into account realistic characteristics of the surface (ε_0 and σ_0 are finite constants). In this case the tangential components of the electromagnetic field on the boundary are continuous and therefore the electromagnetic field must be calculated both in air and in the conductor, which leads to a significant complication of the problem and to a large expense of computer time. On the basis of the results of [3], it can be concluded that a finite electrical conductivity of the conductor leads to a decrease of the amplitude of the radio signal by not more than a factor of two. This can also be understood from simple physical arguments: When $\sigma_0 = \infty$, the radiating electric current is twice as large (because of reflection from the ideally conducting surface) as it is in the case when the surface is a dielectric.

Hence we conclude that the problem can be solved by a simpler method, namely, an expansion in a small parameter depending on the conductivity of the surface σ_0, where the solution in the zero-order approximation is to reduce to the results of [2]. The purpose of the present paper is to work out a general method of numerical solution of the time-dependent problems of generation of radio waves by a gamma-ray source on the boundary of a conducting halfspace and also to obtain a particular analytical solution of the problem for the case of a low-power gamma-ray source.

2. A time-dependent gamma-ray source is located directly on the conducting surface and is switched on at the instant of time t = 0. We choose spherical coordinates (r, φ, z) with the origin at the center of the source and the polar axis along the normal to the surface. Instead of the absolute time t, we introduce the local time $\tau = t - r/c$, where c is the velocity of light. Because the problem has axial symmetry, Maxwell's equations in air can be written in the form

$$\begin{gathered}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\varphi) = \frac{1}{c}\frac{\partial E_r}{\partial\tau} + \frac{4\pi}{c}[\sigma(r,\tau)E_r + j(r,\tau)],\\ -\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) + \frac{1}{c}\frac{\partial B_\varphi}{\partial\tau} = \frac{1}{c}\frac{\partial E_\theta}{\partial\tau} + \frac{4\pi}{c}\sigma(r,\tau)E_\theta,\\ \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{c}\frac{\partial E_\theta}{\partial\tau} - \frac{1}{r}\frac{\partial E_r}{\partial\theta} = -\frac{1}{c}\frac{\partial B_\varphi}{\partial\tau},\end{gathered} \tag{2.1}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 24-29, January-February, 1986. Original article submitted October 25, 1984.

 0021-8944/86/2701-0020$12.50

where $j(r, \tau)$ is the current of Compton electrons and $\sigma(r, \tau)$ is the conductivity of the ionized air. The components of the electromagnetic field must satisfy the initial conditions $E_r = E_\theta = B_\varphi = 0$ for $\tau < 0$. Because the gamma-ray source is surrounded by a perfectly conducting sphere of radius r_0, we have the boundary condition $E_\theta(r = r_\theta, \tau, \theta) = 0$. The region of generation of the electromagnetic field, where there exists a radiating current $j + \sigma E_r$, is bounded by a sphere of radius $r = a$ $(a > r_0)$.

In order to solve the external electromagnetic problem we use the Leontovich boundary condition. Then the field inside the conducting halfspace can be disregarded completely. According to [5], the boundary condition on the Fourier components of the field at $\theta = \pi/2$ is

$$E_r(r, \omega) = \zeta(\omega) B_\varphi(r, \omega), \tag{2.2}$$

where $\zeta(\omega) = (\varepsilon_0 - 4\pi i \sigma_0/\omega)^{-1/2}$ is the surface impedance and we must assume the restriction $|\zeta(\omega)| \ll 1$. We note that the impedance depends on the shape of the surface. Because the characteristic frequencies of the radio emission lie in the frequency region below $\omega_1 \gtrsim 1$ MHz [3], we can assume (following [6]) that in this frequency region the electrical parameters of the surface do not depend on ω and vary from $\varepsilon_0 = 3$, $\sigma_0 = 10^5$ sec^{-1} for dry soil to $\varepsilon_0 = 75$, $\sigma_0 = 5 \cdot 10^{10}$ sec^{-1} for sea water.

The Fourier method cannot be used to solve the system of equations (2.1) and therefore the boundary condition (2.2) must be written in terms of time. As shown in [5], this can be done in the following way:

$$E_r(r, \tau) = \int_0^\tau d\tau' \zeta(\tau - \tau') B_\varphi(r, \tau'), \tag{2.3}$$

and

$$E_r, B_\varphi(r, \tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} E_r, B_\varphi(r, \omega) e^{i\omega\tau}.$$

The surface impedance as a function of time $\zeta(\tau)$ is given by

$$\zeta(\tau) = \frac{1}{2\pi \sqrt{\varepsilon_0}\, \tau} \int_{-x_1}^{x_1} dx \sqrt{\frac{x}{x - ix_0}}\, e^{ix}, \quad x_0 = 4\pi \frac{\sigma_0}{\varepsilon_0} \tau, \quad x = \omega\tau,$$

which can be calculated easily in two cases: $\sqrt{\varepsilon_0} \gg 1$ and $\sqrt{\varepsilon_0} \sim 1$. If $\sqrt{\varepsilon_0} \gg 1$, then we put $x_1 = \infty$ and evaluate the integral with the help of the residue theorem and obtain

$$\zeta_1(\tau/\tau_0) = \frac{1}{\sqrt{\varepsilon_0}\, \tau_0} e^{-\tau/\tau_0} [I_0(\tau/\tau_0) - I_1(\tau/\tau_0)],$$

where $\tau_0 = \varepsilon_0/2\pi\sigma_0$ and I_0 and I_1 are the modified Bessel functions. When $\sqrt{\varepsilon_0} \sim 1$ and $x_1 = \omega_1\tau \ll x_0$, the impedance can be expressed in terms of the Fresnel integrals S and C:

$$\zeta_2(\omega_1\tau) = \frac{1}{(2\pi\tau)^{3/2} \sqrt{\sigma_0}} \left\{ \sqrt{\omega_1\tau}\, (\sin\omega_1\tau + \cos\omega_1\tau) - \sqrt{\frac{\pi}{2}} \left[S\left(\sqrt{\frac{2\omega_1\tau}{\pi}}\right) + C\left(\sqrt{\frac{2\omega_1\tau}{\pi}}\right) \right] \right\}.$$

Graphs of the time dependence of the surface impedance are shown in Fig. 1, where

$$\zeta(x) = \sqrt{\varepsilon_0}\, \tau_0 \zeta_1\left(\frac{\tau}{\tau_0}\right), \quad x = \frac{\tau}{\tau_0}, \text{curve 2}: \zeta(x) = \left(\frac{2\pi}{\omega_1}\right)^{3/2} \sqrt{\sigma_0}\, \zeta_2(\omega_1\tau), \quad x = \omega_1\tau.$$

Now the boundary-value problem (2.1), (2.3) is completely determined with the initial and boundary conditions. We seek a solution in the form of an expansion in the small parameter $\zeta(\tau)/\omega_1 \ll 1$: $E_r, E_\theta, B_\varphi = E_r^0, E_\theta^0, B_\varphi^0 + E_r^1, E_\theta^1, B_\varphi^1 + \ldots$

In the zeroth approximation the boundary condition (2.3) has the form $E_r^0 = 0$ at $\theta = \pi/2$ and the problem reduces to that treated in [2], in which the surface is assumed to be an ideal conductor and the Compton electron current can be written as

$$j = j(r, \tau) \begin{cases} 1, & 0 \leqslant \theta \leqslant \pi/2, \\ -1, & \pi/2 \leqslant \theta \leqslant \pi. \end{cases}$$

Then we expand the Compton current and the field components in series of Legendre polynomials:

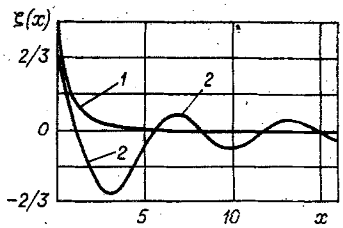

Fig. 1

$$j = j(r,\tau)\sum_{l=1}^{\infty} C_l P_l(\cos\theta),\quad E_r^0 = \sum_{l=1}^{\infty} E_{rl}^0(r,\tau)P_l(\cos\theta),$$

$$E_\theta^0 = \frac{1}{r}\sum_{l=1}^{\infty} E_{\theta l}^0(r,\tau)P_l^1(\cos\theta),\quad B_\varphi^0 = \frac{1}{r}\sum_{l=1}^{\infty} B_{\varphi l}^0(r,\tau)P_l^1(\cos\theta),$$

where $P_l^1(\cos\theta)$ is an associated Legendre polynomial and the coefficients C_l can be expressed in terms of gamma functions

$$C_l = [1-(-1)^l]\frac{2l+1}{4}\sqrt{\pi}/\Gamma\left(\frac{2-l}{2}\right)\Gamma\left(\frac{3l+1}{4}\right).$$

The expansion coefficients $E_{rl}^0, E_{\theta l}^0, B_{\varphi l}^0$ are calculated by numerically integrating a system of one-dimensional time-dependent equations (see [2]).

After solving the problem in the zero-order approximation, it is a simple matter to calculate the i-th approximation using the known solution for the (i – 1)-st approximation, where i = 1, 2,.... The boundary condition (2.3) in this case δ is

$$E_r^i(r,\tau) = \int_0^\tau d\tau'\zeta(\tau-\tau')B_\varphi^{i-1}(r,\tau') \text{ when } \theta = \pi/2.$$

In place of the components of the electromagnetic field E_r^i, we introduce the new function $F_r^i(r,\tau,\theta) = E_r^i(r,\tau,\theta) - \int_0^\tau d\tau'\zeta(\tau-\tau')B_\varphi^{i-1}\left(r,\tau',\theta=\frac{\pi}{2}\right)$. Then Maxwell's equations can be written in the form

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\varphi^i) = \frac{1}{c}\frac{\partial F_r^i}{\partial\tau} + \frac{4\pi}{c}\left[\sigma(r,\tau)F_r^i + j^i(r,\tau)\right],$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi^i) + \frac{1}{c}\frac{\partial B_\varphi^i}{\partial\tau} = \frac{1}{c}\frac{\partial E_\theta^i}{\partial\tau} + \frac{4\pi}{c}\sigma(r,\tau)E_\theta^i,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta^i) - \frac{1}{c}\frac{\partial E_\theta^i}{\partial\tau} - \frac{1}{r}\frac{\partial F_r^i}{\partial\theta} = -\frac{1}{c}\frac{\partial B_\varphi^i}{\partial\tau},$$

where $j^i(r,\tau) = \left[\sigma(r,\tau) + \frac{1}{4\pi}\frac{\partial}{\partial\tau}\right]\int_0^\tau d\tau'\zeta(\tau-\tau')B_\varphi^{i-1}(r,\tau')$, $r \leqslant a$, and $j^i(r,\tau) = 0$ if $r > a$.

The function F_r^i satisfies the boundary condition $F_r^i = 0$ at $\theta = \pi/2$. Then the boundary-value problem in the i-th approximation is analogous to that in the zeroth approximation with the replacement of the quantities $j, E_r^0, E_\theta^0, B_\varphi^0$ by $j^i, F_r^i, E_\theta^i, B_\varphi^i$, respectively. Therefore the solution of the problem in the i-th approximation can be worked out according to the scheme discussed above.

The general method considered here of the numerical solution for the electromagnetic field of a gamma-ray source located on the boundary of air with a conducting halfspace differs from the method used in [3]. First, as a zero-order approximation we choose the known solution of [2] in our method, since this solution is a good approximation to the actual results, particularly when the conductivity of the surface is close to ideal. Secondly, the two-dimensional spatial problem reduces to a set of one-dimensional problems when the Leontovich boundary conditions are used. Thirdly, the solution given here can be easily generalized to the case when the impedance of the halfspace depends on the surface distribution of the parameters ε_0 and σ_0.

3. The problem can be solved analytically in the special case when the conduction current is much smaller than the Compton current. This requirement will be satisfied for a low-power gamma-ray source [7].

We put $\sigma = 0$ in the system of equations (2.1). Then we have the following equations for the Fourier components of the electromagnetic field E_r, E_θ, $B_\varphi(r, \omega, \theta)$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\varphi) = \frac{i\omega}{c}E_r + \frac{4\pi}{c}j(r, \omega),$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) = \frac{i\omega}{c}(E_\theta - B_\varphi), \quad \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial\theta} = \frac{i\omega}{c}(E_\theta - B_\varphi).$$

The boundary conditions are $E_\theta = 0$ at $r = r_0$, $E_r = \zeta(\omega)B_\varphi$ at $\theta = \pi/2$, and $E_r(\tau)$, $E_\theta(\tau)$, $B_\varphi(\tau) = 0$ for $\tau < 0$. We seek a solution to this boundary-value problem in the form of an expansion in the small parameter $|\zeta(\omega)| \ll 1$. The n-th approximation ($n = 0, 1, \ldots$) is calculated according to the method described above. We obtain the following equation for the Fourier components of the magnetic field $B^n_{\varphi l} = B^n_l e^{ikr}$, where $k = \omega/c$:

$$r^2\frac{d^2B^n_l}{dr^2} + [(kr)^2 - l(l+1)]B^n_l = -\frac{4\pi}{c}C_l j^n(r, \omega)e^{-ikr}, \tag{3.1}$$

where

$$j^n(r, \omega) = \begin{cases} j(r, \omega), & n = 0, \\ ik\frac{c}{4\pi}\zeta(\omega)B^{n-1}_\varphi\left(r, \omega, \theta = \frac{\pi}{2}\right), & n = 1, 2, \ldots, \quad r \leqslant a, \\ 0, & n = 1, 2, \ldots, \quad r > a. \end{cases}$$

The functions B^n_l must satisfy the boundary condition $dB^n_l/dr = 0$ at $r = r_0$. Equation (3.1) has a unique solution which can be found with the help of the solution of the homogeneous equation [8]:

$$B^n_l(r, \omega) = \frac{4\pi}{c}C_l r^{l+1}\left[\left(\frac{1}{kr}\frac{d}{dr}\right)^{l+1}e^{ikr}\right]\int_{r_1}^{r}dy\left\{y^{l+1}\left[\left(\frac{1}{ky}\frac{d}{dy}\right)^{l+1}e^{iky}\right]\right\}^{-2}\int_y^{r_2}dx j^n(x, \omega)e^{-ikx}x^{l+1}\left[\left(\frac{1}{kx}\frac{d}{dx}\right)^{l+1}e^{ikx}\right]. \tag{3.2}$$

The parameter r_1 can be expressed in terms of r_0 from the relation $dB^n_l/dr\ (r = r_0) = 0$ and r_2 is determined from the condition that $B_\varphi(\tau)$ be zero for $\tau < 0$. The components of the electromagnetic field in the n-th approximation are written in terms of the functions $B^n_l(r, \omega)$ as

$$B^n_\varphi(r, \omega, \theta) = \frac{e^{ikr}}{r}\sum_{l=1}^{\infty}B^n_l(r, \omega)P^1_l(\cos\theta), \tag{3.3}$$

$$E^n_\theta(r, \omega, \theta) = i\frac{e^{ikr}}{kr}\sum_{l=1}^{\infty}\frac{d}{dr}B^n_l(r, \omega)P^1_l(\cos\theta), \tag{3.3}$$

$$E^n_r(r, \omega, \theta) = -ike^{ikr}\sum_{l=1}^{\infty}\left(1 + \frac{1}{k^2}\frac{d^2}{dr^2}\right)B^n_l(r, \omega)P_l(\cos\theta) + \zeta(\omega)B^{n-1}_\varphi(r, \omega, \theta = \pi/2),$$

where $B^{-1}_\varphi \equiv 0$.

If the electric current is a step function [9]

$$j(r, \tau) = \begin{cases} \frac{\text{const}}{r^2}, & \tau > 0, \\ 0, & \tau \leqslant 0, \end{cases}$$

then in the zeroth approximation ($\sigma_0 = \infty$) and for $l = 1$, the expression calculated from (3.2) and (3.3) reduces for the results of [9], which were obtained by a different analytical method.

The components of the electromagnetic field B^0_φ and E^0_θ can be calculated easily for the simple model where $j(r, \tau) = j_0\delta(\tau)/r^2$. For $l = 1$ we have on the surface ($\theta = \pi/2$)

$$B^0_\varphi\ (r, \tau) = 3\pi\frac{j_0}{rr_0}\left[\left(1 - \frac{r_0}{r}\right)\cos\left(\frac{\sqrt{3}}{2}\frac{c\tau}{r_0}\right) + \frac{1}{\sqrt{3}}\left(1 + \frac{r_0}{r}\right)\sin\left(\frac{\sqrt{3}}{2}\frac{c\tau}{r_0}\right)\right]\exp\left(-\frac{c\tau}{2r_0}\right),$$

$$E_\theta^0(r,\tau)=3\pi\frac{j_0}{rr_0}\left(1-\frac{r_0}{r}\right)\left[\cos\left(\frac{\sqrt{3}}{2}\frac{c\tau}{r_0}\right)+\frac{1}{\sqrt{3}}\left(1+\frac{2r_0}{r}\right)\sin\left(\frac{\sqrt{3}}{2}\frac{c\tau}{r_0}\right)\right]\exp\left(-\frac{c\tau}{2r_0}\right),\ \tau\geqslant 0,\ r\geqslant r_0>0.$$

The radial component $E_r^1(\theta = \pi/2)$ is calculated from (2.3) with the use of $\sqrt{\varepsilon_0} \gg 1$ and $\tau_0 \ll r_0/c$. Then the integral can be evaluated with the help of [10]:

$$E_r^1(r,\tau)=\frac{1}{\sqrt{\varepsilon_0}}B_\varphi^0(r,\tau)\left[1-e^{-\tau/\tau_0}I_0(\tau/\tau_0)\right].$$

The results obtained here can be used to design a numerical calculation of the electromagnetic radiation and to calculate more accurately the parameters of the radio signal from a low-power gamma-ray source.

LITERATURE CITED

1. Yu. A. Medvedev, B. M. Stepanov, and G. V. Fedorovich, The Physics of the Radiative Excitation of Electromagnetic Fields [in Russian], Atomizdat, Moscow (1980).
2. G. G. Vilenskaya, V. S. Imshennik, et al., "Electromagnetic field excited in air by a nonstationary gamma-ray source on an ideally conducting plane," Zh. Prikl. Mekh. Tekh. Fiz., No. 3 (1975).
3. V. Ya. Arsenin, A. A. Dumova, et al., "On the electromagnetic radiation from a nonstationary gamma-ray source on a surface with a finite electrical conductivity," Zh. Vychisl. Mat. Mat. Fiz., 21, No. 2 (1981).
4. V. Ya. Arsenin, V. A. Gorbunov, et al., "On the calculation of the electromagnetic field from a nonstationary gamma-ray source in a self-consistent model," Zh. Vychisl. Mat. Mat. Fiz., 22, No. 2 (1982).
5. L. D. Landau and E. M. Lifshits, Electrodynamics of Continuous Media [in Russian], Nauka, Moscow (1982).
6. G. I. Grudinskaya, Propagation of Radio Waves [in Russian], Vysshaya Shkola, Moscow (1975).
7. V. N. Barkovskii, S. V. Bozhokin, et al., "Generation of a radio signal by a low-power gamma-ray source in the atmosphere," Izv. Vyssh. Uchebn. Zaved., Radiofiz., 25, No. 9 (1982).
8. E. Kamke, Handbook of Ordinary Differential Equations [in Russian], Nauka, Moscow (1971).
9. N. I. Kozlov, "Exact solution of a problem on the propagation of a radio pulse in a nonconducting medium," in: Numerical Methods of Solution of the Problems of Mathematical Physics [in Russian], Nauka, Moscow (1966).
10. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series. Special Functions [in Russian], Nauka, Moscow (1983).